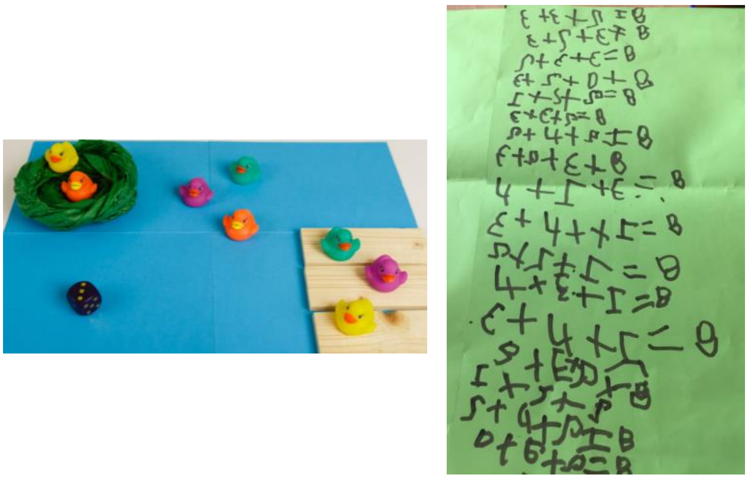
Developing Mathematical Graphics in the Early Years

Janine Davenall, Ann Dowker, Helen J Williams, Catherine Gripton & Sue Gifford

At the Early Childhood Mathematics Group (ECMG), we are often asked by practitioners what emergent mathematical mark-making looks like. This guidance is our response. The questions we consider here are:

1. Why value children's graphical representations and how do they assist children on their mathematical journey?
2. What does emergent mathematical recording look like and how might we support its development?
3. When should formal recording (numerals and symbols) begin?
4. How might mathematical mark-making influence deep-level learning in the early years?

5 frogs on the log or in the pool: 3 and 4 year olds



8 ducks positioned in three places: 5 year old
(photo and activity: Griffiths et al, 2016; Lisa Dandridge, Roseland Primary, Paignton)

1. Why value children’s graphical representations and how do they assist children on their mathematical journey?

Personal recording is important in mathematics at all ages and stages, as it helps us organise or track our thoughts and thus *contributes to our mathematical thinking* and our ability to solve problems. It is also a *means of communicating mathematically* with others. The foundations of both these roles for mathematical mark-making are established when children are young, by adults encouraging them to make jottings alongside their play with practical apparatus, and as we respond positively and with curiosity to children’s attempts to sense-make on paper.

In the early stages, children make deliberate marks as they enjoy experimenting with different tools, surfaces and media. They explore the cause and effect of their actions, trialling different actions to create a range of marks.
For very young children, this can be indistinguishable from emergent writing in that the meaning that the child associates with their marks may be quantities, shape, pattern, words, actions or images. As mark making develops, the distinction between which marks the child intends to represent words and those which have a more mathematical meaning becomes clearer to the adults who know them well and ‘marks’ at the later stages include symbols and drawings.

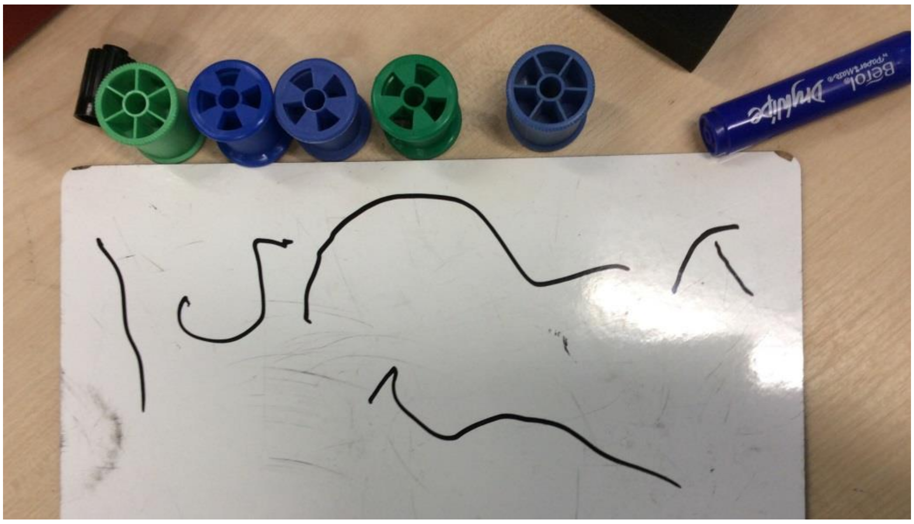


Figure 1. A child’s mathematical graphic

These early marks are often referred to as 'mathematical graphics' (a term popularised by Carruthers & Worthington, 2003). In mathematical mark making the number of marks can be significant as well as their shape, position and orientation, as shown by the example in figure 1. The marks act as a record to 'hold' thoughts as we engage in mathematics. This is as true for adults as for children. The process is therefore crucial. A final record of 4 balls in a basket may show four lines, for example, but the process might have been two then one more then another. This mode of mathematical communication is owned and made sense of by the child. Sometimes mathematical mark-making is not approached as integral to the task, but rather as an end point, as a record of the activity or even the object of the mathematics (e.g. completing a page of 'sums') and this position paper explains how informal mark-making can be used whilst familiarising children with numerals and the generally understood code of mathematics (e.g. introducing and writing numerals) to enhance and deepen children's mathematical development. Informal mathematical mark-making is a key way of children externalising their own internal meaning-making to share with families, practitioners and other children. It is crucial to early mathematical development and invaluable to adults in understanding the child's mathematical thinking. Thus this is a two-way process. It is to do with communication and relationships. Children are apprenticed into the way numbers are culturally represented, as well as developing a repertoire of ways of expressing themselves. Number symbols and signs are positively useful because they are generally understood and help us to communicate mathematically.

Here is the Educational Programme for Mathematics from the Statutory EYFS Framework in England (DfE, 2020):

Developing a strong grounding in number is essential so that all children develop the necessary building blocks to excel mathematically. Children should be able to count confidently, develop a deep understanding of the numbers to 10, the relationships between them and the patterns within those numbers. By providing frequent and varied opportunities to build and apply this understanding - such as using manipulatives, including small pebbles and tens frames for organising counting - children will develop a secure base of knowledge and vocabulary from which mastery of mathematics is built. (DfE 2020:10)

Whilst the terms 'mathematical graphics' or 'mark-making' are not included in the Educational Programme, these are an important part of the '*frequent and varied opportunities to build and apply this understanding'*. This is apparent in the non-statutory guidance, where mathematical mark-making features in both non-statutory EYFS guidance documents (DfE 2021, EE 2021). *Birth to Five Matters* (EE 2021) in particular, points out the connection between mathematical mark-making and mathematical thinking:

Range 3 - *Children should freely explore how they represent their mathematical thinking through gesture, talk, manipulation of objects and their graphical signs and representations, supported by access to graphic tools in their pretend play* (p46 Learning and Development)
Range 4 *- Encourage children to use marks to represent their mathematical ideas in role play* (p97)
Range 5 *- Model and encourage counting and representing numbers within role play, e.g. making a telephone call using a list of numbers. Value children's own mathematical representations within their pretend play. Encourage children to use their fingers to show an amount e.g. when asking another child to share resources, to show on their fingers how many they need.*
Range 6 - *Talk to children about the marks and signs they use to represent and communicate their thinking. As appropriate, model and discuss informal and standard ways (e.g. using arrows, plus and minus signs). Begin to model calculations in mathematical stories and number rhymes and in real contexts, using a range of ways of representing (e.g. five-frames). Use both informal and standard ways to record these, including tallies and symbols. Discuss children's own graphical strategies to solve problems, using some vocabulary of addition and subtraction.*

'Development Matters' (DfE 2021) also values mathematical mark-making:

3 & 4-year-olds will be learning to:
Experiment with their own symbols and marks as well as numerals.
Examples - Encourage children in their own ways of recording (for example) how many balls they managed to throw through the hoop. Provide numerals nearby for reference (p51).

Maths in the early years involves practical experiences, and it is these which children want to represent, manipulatives, structured or unstructured, play an important part of these experiences (Griffiths, Back and Gifford 2016). Manipulating equipment and toys helps to organise our personal ideas in a way that makes sense to ourselves. How might we link the use of manipulatives, as in figure 2, with graphical representations?

Figure 2. Using manipulatives to count

2. What does emergent mathematical recording look like and how might we support its development?

Research is clear that understanding formal representations of mathematics is not easy for young learners, but in the words of Martin Hughes:
"… we can help children build meaningful links between the world of written symbols and the world of concrete reality." (Hughes, 1986: 134)

The work of Martin Hughes in the 1980s brought the importance of informal recording for young learners into focus for us as educators (Hughes, 1986). Hughes researched how 3–7-year-olds recorded quantities and the operations of subtraction and addition. He found that children who had been taught both standard numerals and addition and subtraction symbols and equations often failed to use these when asked to show on paper a task that involved a representing quantity or a change in quantity. Hughes' famous 'box task' and 'tins game' required young children to record the number of bricks hidden inside a box. He classified children's responses under the following four headings:

1) *Idiosyncratic* – where responses did not seem related to the number of objects present.
2) *Pictographic* – representations related to the appearance of what was in front of them as well as numerosity.
3) *Iconic* – representations showing one-to-one correspondence with the number of objects unrelated to the appearance of the objects.
4) *Symbolic* – using conventional symbols to represent each quantity.

(Hughes, 1986: 56-60)

Figure 3 shows some children's responses to a version of Hughes' task where they have been invited to record the number of beans hidden under a pot. As you can see there are a range of responses to this invitation, which we can organise under Hughes' headings:

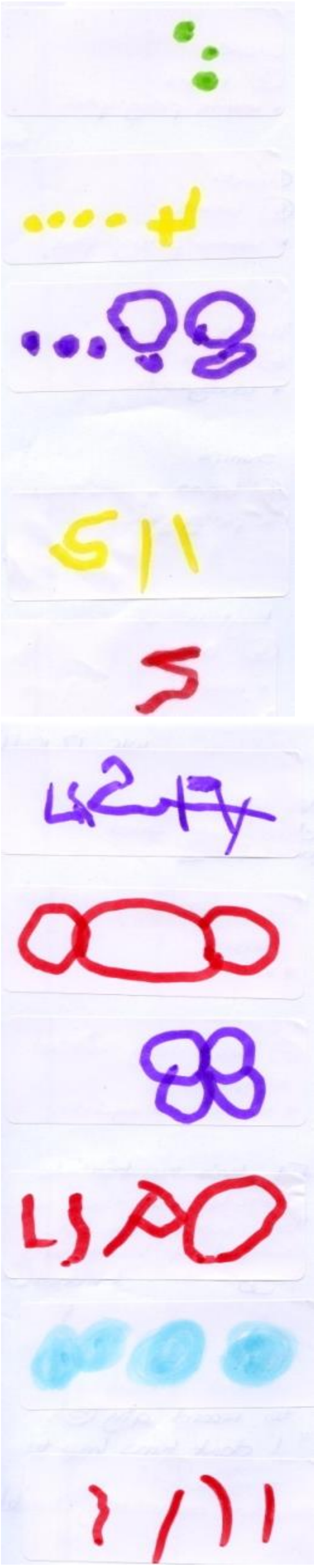


Figure 3. Children's recording of the number of beans under a pot

Hughes' work inspired a number of studies into children's numerical representations. For example, Ewers-Rogers (2002) gave English, Japanese and Swedish children Hughes' box task and obtained similar results, across the cultures, to those of Hughes. She also gave 4 year olds a task that involved writing a note to a milkman to indicate how many milk cartons were wanted. Although the task seems more socially relevant, it appears this was even more difficult for the children. About half of the responses were idiosyncratic without obvious numerical relevance, demonstrating that each task brings its own challenges. Maybe these children by being asked to "write a note" did not consider numerals as part of this communication?

Mathematical ideas need to make *personal sense.* Children's own graphics help them to understand the written language of mathematics and how it can be used (DCSF, 2008). Mathematical graphics offer a conceptual link between practical exploration and symbolic representation. Encouraging children to talk about their recording allows them to rehearse their ideas and provides practitioners with an insight into individuals' mathematical understanding. Fosnot and Dolk (2001) suggest that the representations that children make on a 2D platform are not a copy of what they see but a representation of "their interactions with the object." (2001:78). The lines drawn communicate what is known about an object physically, translated to a 2-dimensional status. So, a tree drawn as several touching circles indicates the experience of walking around it, touching the bark, feeling the shade of the leaves above. A child draws a clock with lines and dots saying, 'tick-tock' as he does so representing the sound experienced. In order to understand the child's expression of their interactions with an object, we therefore need to observe closely. We need to watch and listen as children create graphics and ask them to interpret them for us. Children, as the experts in their own thinking, are able to help us as adults to understand their mathematical thinking through their graphics.

Children's recordings are a means of communication, where emergent mathematical thinking is expressed. As shown by the work of Hughes (1986), we can see how children use marks to represent quantities. Linked to practical, problem-solving contexts such as keeping scores in a game or sharing out an amount of gold for pirates, these reveal children's emerging ideas of applying number and calculation and allow practitioners to support their understanding of standardised symbols. They are permanent records of their thinking and as such can be used to encourage children to reflect on mathematical ideas and new information.

Figure 4. Drawing a hand to represent 'taking away'

In Figure 4, a Reception child has represented 'take away' as the action of a hand:

"*A hand was drawn to symbolise the action of both cars moving away. When prompted, numerals were confidently written for the original and final amounts (2 reversed and zero).*" (Davenall, 2016)

The dynamic element of removing is very evident in this piece of work, something the formal symbols do not convey by themselves.

Talking with children about their graphics often leads to gesture and action as children explain how they combined two quantities, for example. This is particularly powerful where the dramatic action of a story holds significant value for children leading to heightened emotion and energy in the children's explanations. Contexts that are meaningful for the child (part of their world) are powerful. Story contexts can add meaning for the child and are a powerful context for children's mathematics which they express through their graphics.

Figure 5. How many hens are left when the naughty fox sneaks in and eats some?

The examples in figure 5 are Reception children's recording of simple subtractions that were set in a context where the fox was stealing hens. In this example, children were constructing and representing stories based on a real incident, not just finding an answer to a given problem and this of course would add to the rich detail. They were given fluffy chicks to collect and draw, together with a toy fox to enact their story. The children represented their number of chickens and chose the number eaten as well as finding 'How many were left?'. Here we see the subtraction represented by crossing out, an arrow and lines for each hen taken away, as well as the subtraction symbol used conventionally. The children have conveyed so much more than the numerical relationship. They have represented the process (as they perceive it) and the outcome. This is meaningful for the children. Subtraction is not a task with a correct answer but a dynamic situation. Conceptual understanding is developed through the focus on process and these children demonstrate their understanding of subtraction through their graphics.

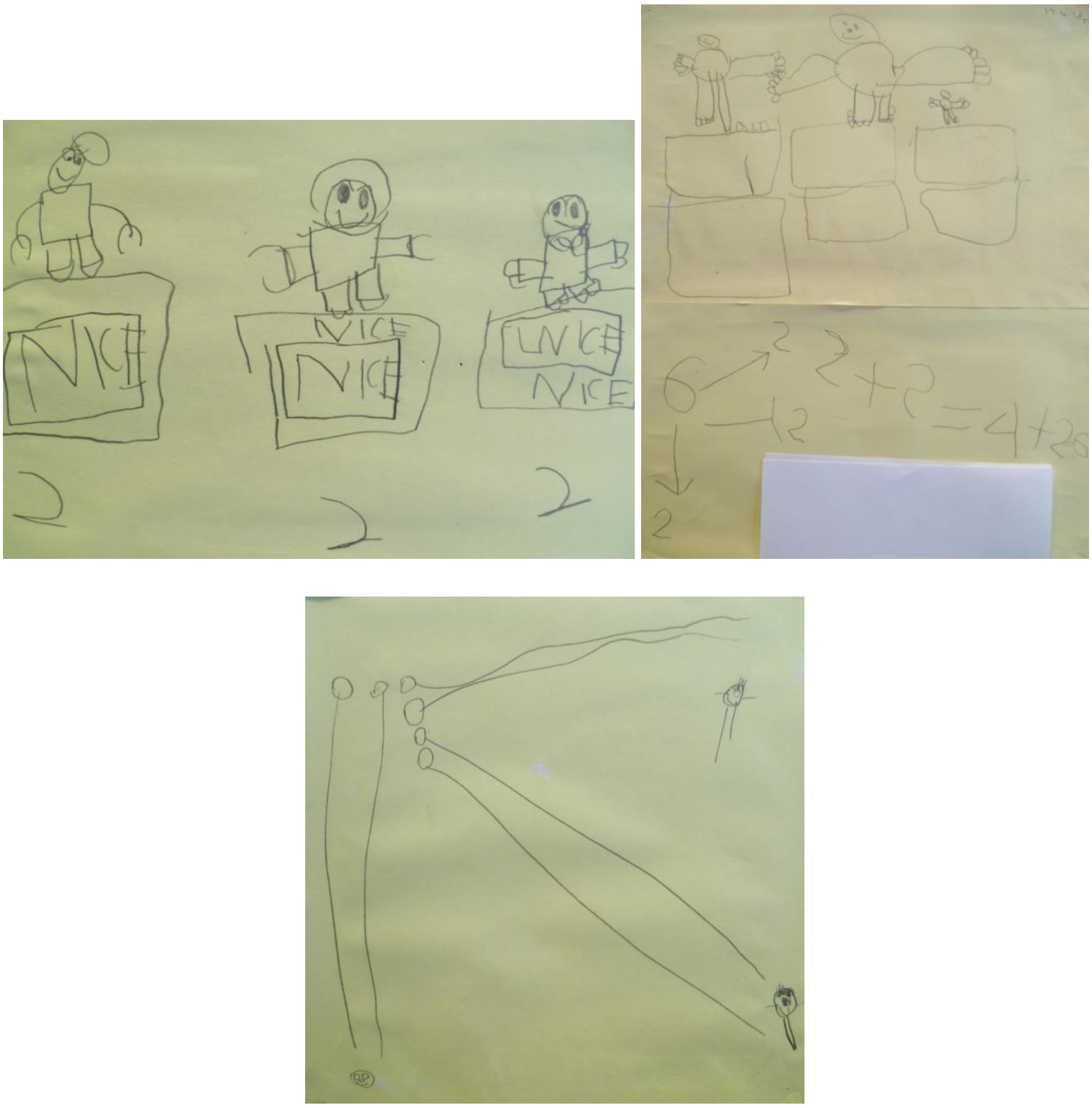


Figure 6. If there are 6 biscuits to be shared fairly between 3 children, how many biscuits will each child receive?

The problem in figure 6 is how to share six biscuits fairly between three children. The children had been given the biscuits to move around and figure 5 shows three children's solutions. It is clear that all three children were able to successfully solve the problem but the graphics show us so much more than this. They show the splitting of 6 into 4 and 2 as well as 2, 2 and 2. They also show us the action of sharing in division and concepts of fair sharing with equal piles or numbers of lines.

Pretend play is a similarly rich context for mathematical mark-making. For example, finding ways of writing price labels for shop items and till receipts for items sold is a purposeful and meaningful context. Allowing children to price objects themselves reveals a lot about their views of value. In Figure 7, a Y1 child is in charge of the Charity Shop and has priced items in whole pounds, allowing them to successfully manage the addition of two or three items. The shop-keeper can then record these as a receipt for the happy purchaser.

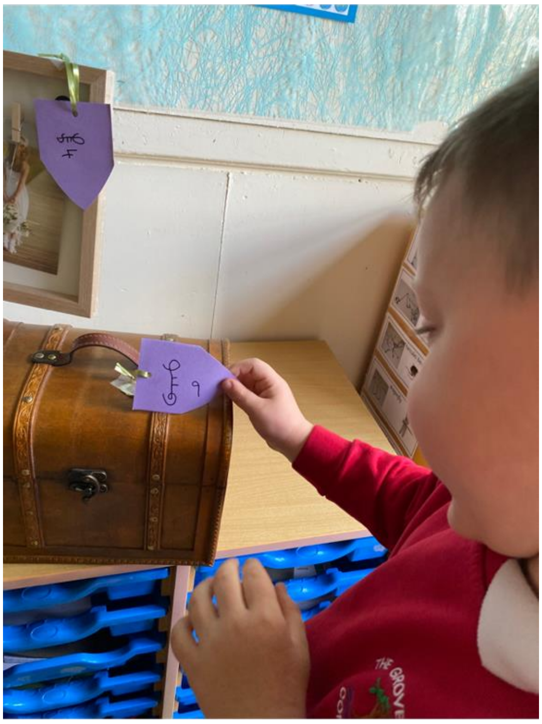


Figure 7. Bargains in a charity shop

Whilst many of the examples in this paper relate to number, mathematical development is supported by mark making in all areas of mathematics. Recording a journey or a route, for example, is rich in communicating children's mathematical thinking (for examples of early map-making, see our spatial reasoning guidance). Children's understanding of shape and perspective is often extended and revealed through their mathematical graphics. If we look at the biscuits in figure 5, we see different ways of communicating a pile of two biscuits (where one is stacked upon the other).

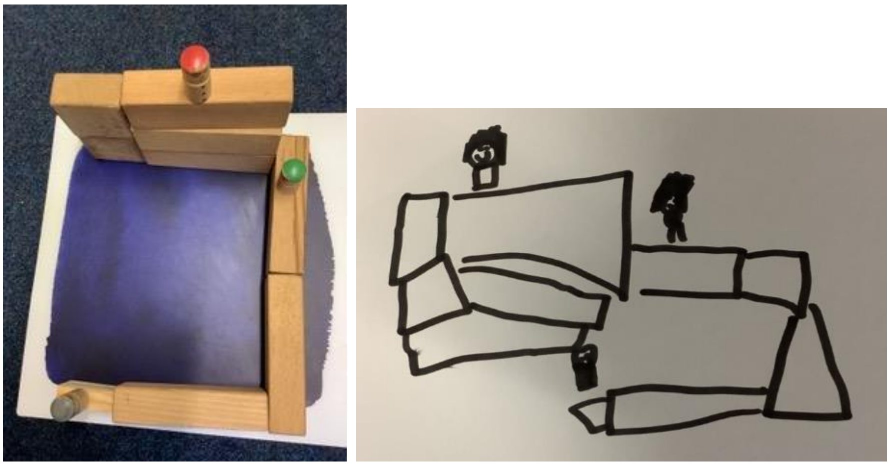


Figure 8: Constructions with small figures: 4 year olds

Children's recording of their block play is fascinating in showing how they perceive units of blocks within their construction and how they represent their three-dimensional model in two dimensions (figure 8).

3. When should formal recording (numerals and symbols) begin?

Worthington and Carruthers (2003) presented evidence that, by working with personalised mathematical graphics, practitioners can support children's development as *"informal marks are gradually transformed into standard symbolism"* (2003:77). This could be one way for practitioners to think about working at 'greater depth' in mathematics. Certainly, it is important for children to learn standard written notation and we obviously expose them to numerals, alongside developing their confidence in making their own mathematical marks, so they can begin to appreciate the benefits of number symbols as efficient tags for amounts. These can be in the form of wooden or plastic numerals, as well as on number cards. This helps them to understand the meaning of the symbols, which is crucial. The move from informal to formal can be seen as a shift in balance, where children continue to use their own jottings and these sit alongside standard ways of representing as their recording gradually adopts standard written forms (Gifford, 1997:76). This takes time and requires a firm foundation rooted in experience. We cannot underestimate the leap of understanding and the depth and range of experience that contributes to a child being able to confidently make one mark, '5', to stand for five separate objects. We also need to help children to learn to write the symbols, which in the case of 5, can be tricky (to start with children tend to reverse numerals, but gradually correct this by the age of 6 or 7).

Using concrete objects as manipulatives is a key aspect of early primary education and forms an important part of preschool mathematics (Clements and Sarama, 2009, 2012). Most 6-year-olds seem to be able to represent at least some addition and subtraction word problems with concrete objects (Carpenter, Hiebert & Moser, 1981; Fuson, 1992; Lindvall & Ibarra, 1980). Younger children appear to have greater difficulty. Dowker (2005) with her students, Mark Gent and Louisa Tate, gave 4, 5 and 6 year old children written (e.g. "2 + 5 = 7"; "6−2 = 4") and orally presented word problems (e.g. "Paul had 4 sweets; his mother gave him 3 more; so now he has 7 sweets"; Alice had 5 buns; she ate 3 buns; so now she has 2 buns". They were asked to show how to find the answer with the counters or "show me the story with the counters". Responses were classified as 'complete' (clearly showing the addition or subtraction); 'incomplete' (showing just the result, or e.g. showing the addends and the result without showing how to get from one to the other) or incorrect (showing the wrong operation; an irrelevant response; or no response. According to the above classification, 44% of the 6-year-olds' responses, 3% of the 5-year-olds' responses and 1% of the 4-year-olds' responses were complete; 28% of the 6-year-olds' responses, 57% of 5-year-olds' responses and 40% of 4-year-olds' responses were incomplete; and 28% of the 6-year-olds' responses, 40% of the 5-year-olds' responses and 60% of the 4-year-olds' responses were incorrect. Thus, the age differences were striking. It is not clear how much of this was due to increasing familiarity with numerals and number words, and how much to their increasing ability to translate between different formats, but this suggests that young children's ability to represent oral and written (formal notation) addition and subtract problems using manipulatives develops considerably between 4 and 6 years of age. It is therefore important to use graphics and manipulatives rather than formal notation or verbal explanation alone for children to at least six years of age, perhaps using formal notation alongside other representations and not moving to formal notation alone too quickly as under 6s may find it challenging to completely understand the problem. Furthermore, work with somewhat older children suggests that, as Hughes' work might indicate, children do not always make spontaneous links between numerals and concrete representations. They may learn most effectively to make the connections if they are simultaneously shown the symbols and concrete representations, and explicitly encouraged to link them (Fuson, 1986; Fuson & Burghardt, 2003; Hart, 1989; Hiebert, J. & Wearne, 1992). Overall, research underlines the huge challenge for young children in making sense of symbolic mathematical representations. As Herbert Ginsberg points out:

> *'If children get off on the wrong symbolic foot, the result may be a nasty fall down the educational stairs.'* Ginsberg (2021)

4. How might mathematical mark-making influence deep-level learning in the early years?

Mathematical mark making supports children to develop effective learning habits or dispositions in mathematics. In creating mathematical graphics, children develop characteristics of effective learning, defined in England (DfE, 2020) as they:

- '*are willing to have a go'* as their mark making is voluntary and low stakes
- *'are involved and concentrating'* showing high levels of involvement and sustaining this concentration on a mathematical activity for longer because of the process involved in mark marking
- *'have their own ideas'* as they determine the mathematics in the situation and what is important to record
- *'choose ways to do things'* as they make decisions about what symbols or marks to use and how to arrange these

- *'find new ways of doing things'* by re-presenting a mathematical situation in a new way and in a way that might be different to how others choose to represent it (including adults). New ways might include more efficient ways.
- *'enjoy achieving what they set out to do'* with ownership over the process and product, they are experts in their own representations

This depth of understanding provides a strong foundation for future mathematical learning in three ways. Firstly, children build new knowledge by connecting and adapting existing understanding to deep understanding, where children can use and apply knowledge flexibly, providing a secure and stable basis for future learning. Secondly, deep understanding gained through mark making enables children to work autonomously, secure in the knowledge that they can record their process in a way that is meaningful to them. This allows them to check and track back, promoting confidence and resilience. It helps them to feel like mathematicians, as insiders not outsiders in the world of school mathematics. Thirdly, mathematical mark-making enables adults to access children's mathematical thinking, in a similar way to manipulatives in that there is a record if thinking that can be shared and discussed. Early valuing of mark making makes children more willing to show their working as they get older. Making mathematical thinking visible in this way enables their teachers to pinpoint errors or difficulties and crucially see *how* the child has approached a problem, knowing where teaching of more efficient methods is needed.

As children move through school, the balance will shift from informal to formal methods of recording, but informal methods are not redundant. They can be a useful stepping stone to learning a new written algorithm or a supplementary way of either checking an answer or initially making sense of a problem before moving to more formal methods for modelling and solving. Mark making for older children might take the form of jottings: having a space to make such jottings while thinking can reduce the pressure on working memory. It can support organisation of thinking and being systematic, as well as encourage risk-taking to try out methods that the child is less certain about.

Some final thoughts

Making marks to represent numbers, shapes or mathematical operations need not always be on paper. Children have increasing opportunities for mark-making with technology; and these have been used for some time in early mathematics curricula and interventions (e.g. Sarama & Clements, 2021). This may facilitate learning, by removing the need for manual dexterity. Price, Jewitt & Crescenzi (2015) observed 2 and 3 year-old children engaging in a free finger-painting activity and a colouring activity, both on paper with physical paint and on a tablet computer. Videos suggested that the tablet limited the number of fingers used, limited the sensory experience of using paint, and resulted in more uniform final compositions. However, it increased speed and continuity, which resulted in more mark-making and different scales of mark-making. Although this study did not look specifically at mathematical activities, it does suggest that traditional and computer-based mark-making activities may have different advantages to offer and may complement each other.

If we accept that *all* young children are able to think deeply about and be proficient in mathematics, which is the underlying principle at the heart of learning and teaching for mastery, then encouraging and supporting children's own emerging mathematical graphics is one way of enriching and deepening children's understanding. If we wish our teaching to build on what children know and can do mathematically, children's informal mathematical graphics provide a window into their thinking, and one we would do well to take account of.

We invite families and practitioners to send us children's mathematical graphics to accompany this piece. Please add a sentence to explain the context, the age of the children and attach them to an email to admin@earlyyears.org

References

Carpenter, T.P., Hiebert, J. & Moser, J.M. (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*, *18*, 83–97.

Clements, D. H., & Sarama, J. (2021). *Learning and teaching early math: the learning trajectories approach.* 3rd edition. New York: Routledge.

Carruthers, E., & Worthington, M. (2011). *Understanding children's mathematical graphics: beginnings in play.* Maidenhead: McGraw-Hill Education.

Clements, D. H., & Sarama, J. (2012). Mathematics learning, assessment, and curriculum. In R. C. Pianta, L. Justice, S. W. Barnett, & S. Sheridan (Eds.), *Handbook of early education.* New York: Guilford, 217- 239.

Davenall, J. (2016). *Young children's mathematical recording*. Available at:

https://nrich.maths.org/12384

DCSF (2008). *Mark making matters: Young children making meaning in all areas of learning and development.* Available at: https://www.foundationyears.org.uk/wp-content/uploads/2011/10/Mark_Marking_Matters.pdf

DfE (2020). *Statutory Framework for the Early Years Foundation Stage.* Available at: https://www.gov.uk/government/publications/early-years-foundation-stage-framework--2

DfE (2021). *Development Matters: Non-statutory curriculum guidance for the Early Years Foundation Stage*. Available at: https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/988004/Development_Matters.pdf

Dowker, A. (2005). *Individual differences in arithmetic: Implications for psychology, neuroscience and education.* Hove: Psychology Press.

Early Education (2021). *Birth to five matters.* Available at: https://www.birthto5matters.org.uk

Ewers-Rogers, J. (2002). *Very young children's understanding and use of numbers and number symbols*. University of London Institute of Education: Unpublished Ph.D. thesis.

Fosnot, C.T. & Dolk, M. (2001). *Young mathematicians at work: Constructing number sense, addition and subtraction.* Portsmouth NH: Heinneman.

Fuson, K. (1986). Roles of representation and verbalization in the teaching of multi-digit addition and subtraction. *European Journal of the Psychology of Education*, *1*, 35-56.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* New York: Macmillan, 243-275.

Fuson, K. & Burghardt, B. (2003). Multidigit addition and subtraction methods invented in small groups and teacher support of problem solving and reflection. In A. Baroody & A. Dowker (eds.) *The Development of Arithmetic Concepts and Skills*. Mahwah, N.J.: Erlbaum, 267-304.

Gifford, S. (1997). 'When should they start doing sums? A critical consideration of the 'emergent mathematics' approach', in Thompson, I. (Ed.) *Teaching and learning early number*, Buckingham: Open University Press.

Ginsberg, H. (2021). *What children know and need to learn about counting.* Available at: https://prek-math-te.stanford.edu/counting/what-children-know-and-need-learn-about-counting

Griffiths, R., Back, J. and Gifford, S. (2016). *Making numbers: Using manipulatives to teach arithmetic.* Oxford: Oxford University Press.

Hart, K. (1989). There is little connection. In P. Ernest (ed.) *Mathematics teaching: the state of the art*. London: Falmer, 138–142.

Hiebert, J. & Wearne, D. (1992). Links between teaching and learning place value with understanding in first grade. *Journal for Research in Mathematics Education*, *23*, 98-122.

Hughes, M. (1986). *Children and number: Difficulties in learning mathematics.* Oxford: Basil Blackwell.

Lindvall, C.M. & Ibarra, C.G. (1982). Incorrect procedures used by primary grade pupils in solving open addition and subtraction problems. *Journal for Research in Mathematics Education*, *11*, 50–62.

Price, S., Jewitt, C. & Crescenzi, L. (2015). The role of iPads in pre-school children's mark making development. *Computers and Education, 87*, 131-141.

Sarama, J. & Clements, D.H. (2009). "Concrete" computer manipulatives in mathematics education. *Child Development Perspectives*, 3(3), 145-150.

Thompson, I. (Ed) (2008). *Teaching and learning early number.* Second edition. Oxford: OUP.

Worthington, M. & Carruthers, E. (2003). *Children's mathematics: Making marks, making meaning.* London: Paul Chapman Publishing.

Acknowledgments:
Children's work from Janine Davenall, Helen Williams, Rachel Fleming, Sharon Palfreyman